4.3 1st Differentiating Natural Exponentials
4.3 2nd Differentiating Natural Logarithms.

4.3 1st

Objectives

1) Find the derivative of a function involving a natural exponential $\frac{d}{dx}(e^x)$.
   - → chain rule
   - → product rule
   - → quotient rule

2) Recognize the derivative of an exponential is NOT the derivative of a power function

3) Use calculus to graph a natural exponential function.
   - → critical values
   - → increase / decrease
   - → relative extrema
   - → concave up / down
   - → inflection points

4) Recognize derivative as rate of change in an application.

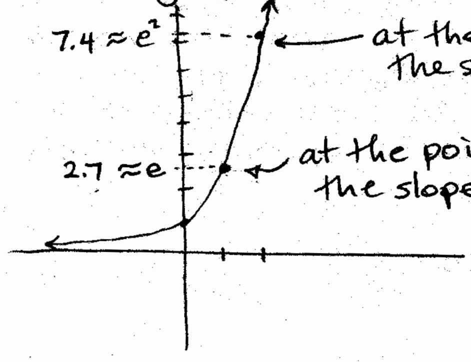
KEY CONCEPT:

$$\frac{d}{dx}(e^x) = e^x$$

The natural exponential is its own derivative!

This means: If we graph $f(x) = e^x$

x	$f(x)=e^x$
-2	$e^{-2} = 1/e^2$
-1	$e^{-1} = 1/e$
0	$e^0 = 1$
1	$e^1 = e$
2	e^2



← at the point $(2, e^2)$ the slope of the tangent line is e^2.

← at the point $(1, e)$ the slope of the tangent line is e.

Find derivatives.

① $f(x) = e^{2x}$

* This is the composition $\left.\begin{array}{l} g(x) = e^{x} \\ h(x) = 2x \end{array}\right\} g(h(x))$

So this is a <u>chain rule</u>!

$f'(x) = e^{2x} \cdot \frac{d}{dx}(\text{inside})$

derivative of "outside" is still evaluated at 2X $\Rightarrow e^{2x}$

$= e^{2x} \cdot \frac{d}{dx}(2x)$

$= e^{2x} \cdot 2$

$= \boxed{2e^{2x}}$

← typically, we write coefficients at front.

CHAIN RULE

$$\frac{d}{dx} e^{g(x)} = g'(x) \cdot e^{g(x)}$$

② $f(x) = \frac{e^{x}}{x^{2}}$

We'll do this by two methods.

option 1: rewrite with negative exponent, use product rule.

$f(x) = x^{-2} \cdot e^{x}$

$f'(x) = \frac{d}{dx}(x^{-2}) \cdot e^{x} + \frac{d}{dx}(e^{x}) \cdot x^{-2}$

$= -2x^{-3}e^{x} + e^{x} \cdot x^{-2}$

$= -2x^{-3}e^{x} + x^{-2}e^{x}$

factor GCF and least powers!

$= e^{x}(-2x^{-3} + x^{-2})$

$-3 < -2$

$= x^{-3} \cdot e^{x} \cdot (-2 + x^{-2-(-3)})$

subtract exp you factor out

$-2-(-3)$

$-2+3$

1 ✓

$= \boxed{x^{-3}e^{x}(-2+x)}$

$= \boxed{\frac{(x-2) \cdot e^{x}}{x^{3}}}$

option 2: Quotient rule

$f(x) = \frac{e^x}{x^2}$

$f'(x) = \frac{x^2 \cdot \frac{d}{dx}(e^x) - e^x \cdot \frac{d}{dx}(x^2)}{(x^2)^2}$ ← Don't forget the denominator!

$= \frac{x^2 \cdot e^x - e^x \cdot 2x}{x^4}$

$= \frac{x^2 e^x - 2x e^x}{x^4}$ factor GCF & least powers

$= \frac{x e^x [x - 2]}{x^4}$ reduce $\frac{x}{x^4}$

$= \boxed{\frac{e^x (x-2)}{x^3}}$

(3) Recall $\frac{d}{dx}(3) = 0$ derivative of a constant.

(4) $f(x) = e$

$\boxed{f'(x) = 0}$

e and e^6 are just constants!

(5) $f(x) = e^6$

$\boxed{f'(x) = 0}$

(6) $f(x) = ex^2$

↑ This e is a constant multiple....

easier example
$f(x) = 3x^2$
$f'(x) = 3 \cdot 2x = 6x$!

$f'(x) = e \cdot 2x$

$\boxed{f'(x) = 2ex}$

(7) $f(t) = \sqrt{e^{2t}+4}$

rewrite

$f(t) = (e^{2t}+4)^{1/2}$ chain rules outermost power $\frac{1}{2}$

$f'(t) = \frac{1}{2}(e^{2t}+4)^{-1/2} \cdot \frac{d}{dt}(e^{2t}+4)$

$= \frac{1}{2}(e^{2t}+4)^{-1/2} \cdot (e^{2t} \cdot \frac{d}{dt}(2t) + 0)$ ← constant term.

↑ chain rule again!

$= \frac{1}{2}(e^{2t}+4)^{-1/2}(e^{2t} \cdot 2)$

$= \dfrac{\frac{1}{2} \cdot 2 \cdot e^{2t}}{(e^{2t}+4)^{1/2}}$

$= \boxed{\dfrac{e^{2t}}{\sqrt{e^{2t}+4}}}$

(8) $f(z) = \dfrac{e^z}{1+e^z}$

Quotient rule

$f'(z) = \dfrac{(1+e^z) \cdot \frac{d}{dz}(e^z) - e^z \cdot \frac{d}{dz}(1+e^z)}{(1+e^z)^2}$

$= \dfrac{(1+e^z) \cdot e^z - e^z \cdot e^z}{(1+e^z)^2}$

distribute e^z

$= \dfrac{e^z + e^z \cdot e^z - e^z \cdot e^z}{(1+e^z)^2}$

$e^z \cdot e^z = e^{2z}$, but it doesn't matter when we combine like terms

$= \boxed{\dfrac{e^z}{(1+e^z)^2}}$

Note: Can also do by rewriting as negative exp ⇒ Product and Chain rules!

(9) $f(z) = \dfrac{12}{1+2e^{-z}}$

HANDWRITING ALERT
$2 \neq z$

Option 1:
Quotient Rule

$$f'(z) = \frac{(1+2e^{-z})\cdot\frac{d}{dz}(12) - 12\cdot\frac{d}{dz}(1+2e^{-z})}{(1+2e^{-z})^2}$$

$$= \frac{0 \quad - 12\left(2e^{-z}(-1)\right)}{(1+2e^{-z})^2}$$

chain rule $\frac{d}{dz}(-z) = -1$

$$= \boxed{\frac{+24e^{-z}}{(1+2e^{-z})^2}}$$

Option 2:
Rewrite with another negative exponent

$f(z) = 12(1+2e^{-z})^{-1}$ chain rules

$$f'(z) = -12(1+2e^{-z})^{-2}\cdot\frac{d}{dz}(1+2e^{-z})$$

$$= -12(1+2e^{-z})^{-2}\left(0+2e^{-z}(-1)\right)$$

chain rule $\frac{d}{dz}(-z) = -1$

$$= -12(1+2e^{-z})^{-2}(-2e^{-z})$$

$$= \boxed{\frac{24e^{-z}}{(1+2e^{-z})^2}}$$

(10) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Quotient Rule

$$f'(x) = \frac{(e^x+e^{-x})\cdot\frac{d}{dx}(e^x-e^{-x}) - (e^x-e^{-x})\cdot\frac{d}{dx}(e^x+e^{-x})}{(e^x+e^{-x})^2}$$

$$= \frac{(e^x+e^{-x})(e^x-e^{-x}(-1)) - (e^x-e^{-x})(e^x+e^{-x}(-1))}{(e^x+e^{-x})^2}$$

$$= \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$ FOIL twice!

$$= \frac{(e^{2x}+e^0+e^0+e^{-2x}) - (e^{2x}-e^0-e^0+e^{-2x})}{(e^x+e^{-x})^2}$$

$$= \frac{(e^{2x}+1+1+e^{-2x}) - (e^{2x}-1-1+e^{-2x})}{(e^x+e^{-x})^2}$$

$$= \frac{e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}}{(e^x+e^{-x})^2}$$

$$= \boxed{\frac{4}{(e^x+e^{-x})^2}}$$

Though we could re-write with negative exponent and use product and chain rules ... there are a lot of signs!

$f(x) = (e^x - e^{-x})(e^x + e^{-x})^{-1}$

(11) Write the equation of the tangent line to $f(x) = x^2 e^{x+1}$ at $x = -1$.

recall: derivative tells us the slope of the tangent line.

$f'(x) = x^2 \cdot \frac{d}{dx}(e^{x+1}) + \frac{d}{dx}(x^2) \cdot e^{x+1}$ product rule

$= x^2 \cdot e^{x+1} \cdot 1 + 2x e^{x+1}$

$f'(x) = x e^{x+1}(x+2)$ factor GCF e^{x+1} least power x

$f'(-1) = -1 e^{-1+1}(-1+2)$

$= -1e^0(1)$

$= -1$

slope $m = -1$

Need y-coordinate, from original function

$f(-1) = (-1)^2 e^{-1+1}$

$= 1e^0$

$= 1$

point $(-1, 1)$

Equation

$y - 1 = -1(x+1)$

$y - 1 = -x - 1$

$\boxed{y = -x}$

or

$y = mx + b$

$1 = -1(-1) + b$

$0 = b$

$\boxed{y = -x}$

(12) Sketch graph of $f(x) = e^{-x^6/6}$ using calculus

- find intervals of increase/decrease
- find relative extrema
- find intervals of concave up/down
- find inflection points.

$$f'(x) = e^{-x^6/6} \cdot \frac{d}{dx}\left(\frac{-x^6}{6}\right)$$ chain rule

$$= e^{-x^6/6} \cdot \frac{-1}{6} \cdot 6x^5$$

$$= -x^5 e^{-x^6/6}$$

$f'(x) = 0$ $\quad \frac{-x^5}{e^{+x^6/6}} = 0$

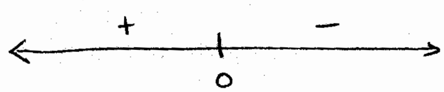
$$-x^5 = 0 \cdot e^{x^6/6}$$

$$-x^5 = 0$$

$$x^5 = 0$$

$x = 0$. c.v. (horizontal tangent line)

$f'(x)$ undefined $\quad e^{x^6/6} = 0$ no solution (no vertical tangent line)

f' 

increasing $(-\infty, 0)$
decreasing $(0, \infty)$
relative max at $x=0$, $y=1$ $\quad f(0) = e^{-0/6} = e^0 = 1$

$$f''(x) = \frac{d}{dx}(-x^5) \cdot e^{-x^6/6} + (-x^5) \cdot \frac{d}{dx}\left(e^{-x^6/6}\right)$$ product rule!

$$= -5x^4 e^{-x^6/6} - x^5 \cdot e^{-x^6/6} \cdot \frac{d}{dx}\left(\frac{-x^6}{6}\right)$$ ← chain rule

$$= -5x^4 e^{-x^6/6} - x^5 \cdot e^{-x^6/6} \cdot \left(-\frac{1}{6}\right) \cdot 6x^5$$

$$= -5x^4 e^{-x^6/6} + x^{10} e^{-x^6/6}$$

$$= x^4 e^{-x^6/6}(-5 + x^6)$$

$$= \frac{x^4 (x^6 - 5)}{e^{x^6/6}}$$

$f''(x)=0 \qquad x^4=0$

$x=0$

$x^6-5=0$

$x^6=5$

$x=\pm\sqrt[6]{5}$ even-index radical requires $\pm$

$f''(x)$ undefined $e^{x^6/6}=0$ no solution

f'': $+$ | $-\sqrt[6]{5}$ | $-$ | 0 | $-$ | $\sqrt[6]{5}$ | $+$

test $x=-2$, test $x=-1$, test $x=1$, test $x=2$

$\sqrt[6]{5}\approx 1.3$

concave up $(-\infty,-\sqrt[6]{5})$, $(\sqrt[6]{5},\infty)$

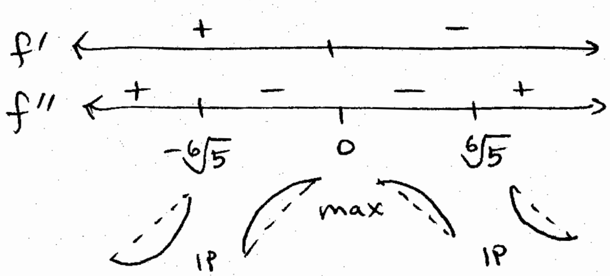
concave down $(-\sqrt[6]{5},\sqrt[6]{5})$

inflection points $(-\sqrt[6]{5},e^{-5/6})$

$(\sqrt[6]{5},e^{-5/6})$

$f(\sqrt[6]{5})=e^{-(\sqrt[6]{5})^6/6}$

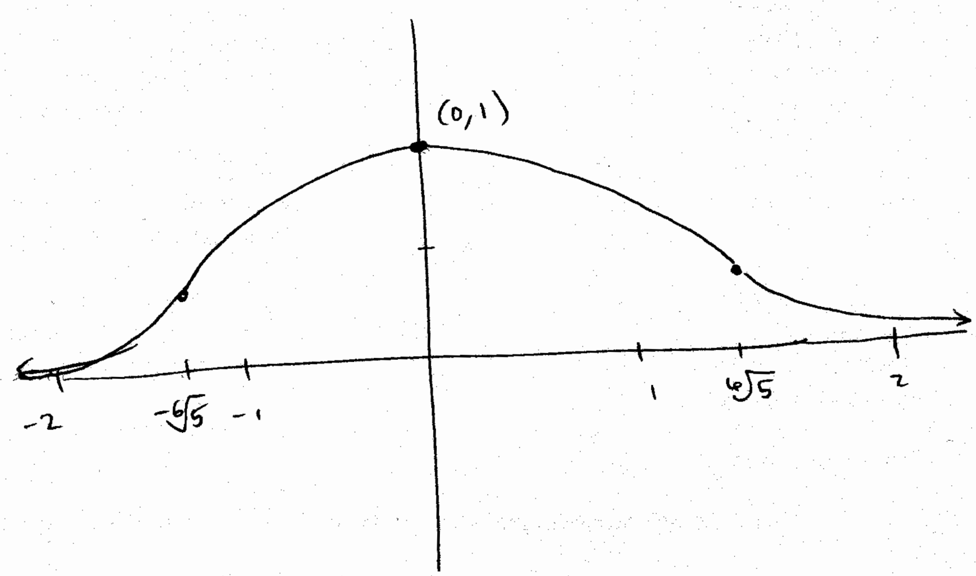
$=e^{-5/6}\approx .43$



max $(0,1)$

IP $(-\sqrt[6]{5},e^{-5/6})\approx(-1.3,.4)$

$(\sqrt[6]{5},e^{-5/6})\approx(1.3,.4)$



(13) Personal Finance

A \$10,000 car depreciates so its value after t years is

$$V(t) = 10{,}000e^{-0.35t} \text{ dollars.}$$

Find the instantaneous rate of change of its value

a) when it is new $(t=0)$

b) after 2 years

Recall: instantaneous rate of change means derivative!

$$V'(t) = 10{,}000 \cdot e^{-0.35t} \cdot \frac{d}{dt}(-0.35t)$$ ← chain rule

$$= 10{,}000e^{-0.35t}(-0.35)$$

$$= -3500e^{-0.35t}$$

a) $$V'(0) = -3500e^{(-0.35 \times 0)}$$

$$= -3500e^{0}$$

$$= -3500 \cdot 1$$

$$= -3500$$

Value is decreasing \$3500/yr at the moment of purchase

b) $$V'(2) = -3500e^{(-0.35 \times 2)}$$

$$= -3500e^{-.7}$$

$$\approx -1738.05$$

Value is decreasing \$1738.05/yr 2 years later